Critical phenomena in networks

A. V. Goltsev,[1,2,*] S. N. Dorogovtsev,[1,2,†] and J. F. F. Mendes[1,3,‡]

[1]*Departamento de Física and Centro de Física do Porto, Faculdade de Ciências, Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal*

[2]*Ioffe Physico-Technical Institute, 194021 St. Petersburg, Russia*

[3]*Departamento de Física, Universidade do Aveiro, Campus Santiago, 3810-193 Aveiro, Portugal*

We develop a phenomenological theory of critical phenomena in networks with an arbitrary distribution of connections $P(k)$. The theory shows that the critical behavior depends in a crucial way on the form of $P(k)$ and differs strongly from the standard mean-field behavior. The critical behavior observed in various networks is analyzed and found to be in agreement with theory.

Many complex interacting systems such as the brain, Internet, social systems, etc., are recognized as networks. They demonstrate a spectrum of unique effects [1–5], which distinguish them from all other known structures of condensed matter. One could expect that such systems have peculiar cooperative phenomena. Only the first studies of particular cooperative models on particular networks were performed recently [6–18] . They were focused on traditional models in statistical physics, such as the Ising and XY models [7–14], percolation [15–17], epidemic spreading, etc., in scale-free networks [18]. It was revealed that their critical behavior is more rich and extremely far from that expected from the standard mean-field theory. A deviation from the usual mean-field critical behavior appears when, depending on the model, the fourth $\langle k^4 \rangle$ or third $\langle k^3 \rangle$ moments of the degree distribution $P(k)$ diverge. Here, *degree* is the number of connections of a vertex. When $\langle k^2 \rangle$ diverges, all models undergo an unusual phase transition of infinite order. This case is of primary importance because many real networks such as the Internet and biological nets are described by $P(k)$ with infinite $\langle k^2 \rangle$.

Why do critical phenomena in networks differ so much from those in usual substrates and what is their common origin? Why do all investigated models demonstrate universal behavior when $\langle k^2 \rangle$ diverges? In order to answer the questions raised above and analyze results of simulations and experiments from a general point of view, it is necessary to have a general theory which is not restricted by specific properties of any model.

The advantage of the phenomenological approach is that the origin of interactions and nature of interacting objects are irrelevant. These can be spins, percolating clusters, biological objects, etc. In this article, we develop a phenomenological theory of cooperative phenomena in networks. Our approach is based on concepts of the Landau theory of continuous phase transitions. It is shown that the critical behavior in networks has a universal character and is determined by two general properties: (i) the structure of a network and (ii) the symmetry underlying a model. We find that in networks described by a degree distribution with divergent moments, the thermodynamic potential of an interacting system with cooperative effects is a singular function of an order parameter. The theory is in complete agreement with exact results for the Ising and Potts models and percolation on "equilibrium" uncorrelated random networks. On the basis of the theory we discuss results of theoretical studies and simulations of the Ising and XY models and epidemic spreading on evolving and small-world networks.

Let us consider a system of interacting objects. Interactions or links between these objects form a net. We assume that some kind of 'order' can emerge. This "ordered" phase may be characterized by some quantitative characteristic x while it will vanish in a "disordered" phase above a critical point. In order to study the critical behavior we assume that the thermodynamic potential Φ of the system is not only a function of the order parameter x but also depends on the degree distribution

$$\Phi(x,H) = -Hx + \sum_{k}^{\infty} P(k)\phi(x,kx). \tag{1}$$

Here H is a field conjugated with x. It should be emphasized that Eq. (1) is not obvious *a priori*. The function $\phi(x,kx)$ may be considered as a contribution of vertices with k connections. There are arguments in favor of this assumption. Let us consider the interaction of an arbitrary vertex 0 with k neighboring vertices. In the framework of a mean-field approach, k neighboring sites having a spontaneous "moment" x produce an effective field kx acting on the site 0. This indicates that the expansion is over not only x but also kx.

We assume that $\phi(x,y)$ is a smooth function of x and y and can be represented as a series in powers of both x and y:

$$\phi(x,y) = \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \phi_{ml}x^m y^l, \tag{2}$$

where ϕ_{ml} are functions of "temperature" T and H. We use the term "temperature" for convenience. If a change of another parameter leads to the emergence of "order," then this control parameter can be used as T.

---

*Email address: goltsev@pop.ioffe.rssi.ru

†Email address: sdorogov@fc.up.pt

‡Email address: jfmendes@fc.up.pt

There are general restrictions on the coefficients ϕ_{ml}. For an arbitrary $P(k)$, at zero field $H=0$, the expansion of $\Phi(x,H=0)$ over x must contain only the second and higher powers of x. Therefore, $\phi_{00}=\phi_{01}=\phi_{10}=0$.

Φ must be finite if $\langle k \rangle$ is finite. This condition is satisfied if at $y \gg 1$ and an arbitrary x the function $\phi(x,y)$ increases slower than y:

$$\phi(x,y) \lesssim g(x)y, \qquad (3)$$

where $g(x)$ is a function of x. Equations (1)–(3) are the basis of our theory.

In the framework of the Landau theory, it is assumed that the thermodynamic potential Φ near a phase transition point can be represented as a series in powers of the order parameter x:

$$\Phi(x,H) = -Hx + f_2 x^2 + f_3 x^3 + f_4 x^4 + \cdots . \qquad (4)$$

Higher terms in the expansion (4) are irrelevant for the critical behavior. The coefficients f_n are related to the coefficients ϕ_{ml} in the expansion (2). Let us suppose first that the critical temperature T_c is finite. (The case of an "infinite" critical temperature will be considered separately.) Following Landau, we assume that near the critical temperature the coefficient f_2 can be written as $a(T-T_c)$ where $a>0$ due to the condition of stability of the disordered phase at $T>T_c$. $f_2=0$ at the transition point. This gives the following equation for T_c:

$$\phi_{20}(T_c) + \phi_{11}(T_c)\langle k \rangle + \phi_{02}(T_c)\langle k^2 \rangle = 0. \qquad (5)$$

According to this equation, in the case $\phi_{20}=0$ the critical temperature T_c depends only on the ratio $\langle k^2 \rangle/\langle k \rangle$. Such a dependence was revealed in all studied models on networks [8,9,15,17,18]. In the ordered phase the order parameter x is determined by the condition that $\Phi(x,H)$ is minimum:

$$d\Phi(x,H)/dx = 0. \qquad (6)$$

Solving Eq. (6), we find $x(T,H)$ and all other physical parameters: the response function $\chi_x = dx/dH$, the specific heat $C = -Td^2\Phi/dT^2$, etc.

The condition of the stability of the ordered phase near the continuous phase transition demands that either $\langle f_3 \rangle > 0$ or if $\langle f_3 \rangle = 0$, then $\langle f_4 \rangle > 0$. In the general case, $f_{2n+1} \neq 0$. If for symmetry reasons $\Phi(x,H) = \Phi(-x,-H)$, then $f_{2n+1} = 0$. This condition takes place if $\phi_{ml} = 0$ at $m+l=2n+1$.

Let us consider analytical properties of Φ in the general case. The nth derivative of $\Phi(x,H)$ at $x=0$ is equal to

$$\Phi^{(n)}(0) \equiv d^n\Phi(x,H)/dx^n|_{x=0} = n! \sum_{l=0}^{n} \phi_{n-l,l}\langle k^l \rangle. \qquad (7)$$

If all the moments $\langle k^l \rangle = \Sigma_k P(k)k^l$ converge, then all the derivatives $\Phi^{(n)}(0)$ are finite. Therefore, $\Phi(x,H)$ is a smooth function of x, at least at small x. However, in most interesting real networks, $P(k)$ has divergent moments. This leads to a very important consequence. If moments $\langle k^l \rangle$ with $l<p$ converge and moments $\langle k^l \rangle$ with $l \geq p$ diverge, then

$\Phi^{(l)}(0)$ with $l<p$ are finite while $\Phi^{(l)}(0) \to \infty$ for $l \geq p$. In this case $\Phi(x,H)$ is a singular function of x and can be written as follows:

$$\Phi(x,H) = -Hx + \sum_{n=2}^{p-1} f_n(k)x^n + x^p s(x), \qquad (8)$$

where $s(x)$ is a singular function, $s(0)$, $s^{(n)}(0) \to \infty$, but $xs(x) \to 0$ at $x \to 0$. This result is of primary importance. It is the singular function that can lead to a deviation from the standard mean-field behavior. A method for determining $s(x)$ is proposed below. For convenience we use a power-law degree distribution $P(k) \propto k^{-\gamma}$. Then, $\langle k^4 \rangle$ diverges for $\gamma \leq 5$, $\langle k^3 \rangle$ diverges for $\gamma \leq 4$, and $\langle k^2 \rangle$ is divergent for $\gamma \leq 3$.

We begin with the case $f_{2n+1}=0$. Let us consider the critical behavior for $f_4 > 0$ and different γ.

(a) $\gamma > 5$. Here, $\langle k^4 \rangle$ converges. The coefficient f_4 is finite and Φ has a usual form (4). At $H=0$ Eq. (6) leads to the standard critical behavior

$$x \propto \tau^{1/2}, \Delta C \neq 0, \chi_x(H=0) \propto \tau^{-1}. \qquad (9)$$

Here, $\tau = 1 - T/T_c$, $\Delta C \sim 1/\langle k^4 \rangle$ is the jump of the specific heat at T_c. It tends to zero when $\gamma \to 5$.

(b) $3 < \gamma \leq 5$. Here, $\langle k^4 \rangle$ diverges. According to Eq. (8), a singular term $x^4 s(x)$ is expected to appear in Φ. The function $\phi(x,y)$ in Eq. (2) may be divided into two parts:

$$\phi(x,y) = \sum_{m=0}^{\infty} \sum_{1 \leq l < \gamma - 1} \phi_{ml}x^m y^l + G(x,y). \qquad (10)$$

The first term determines a nonsingular contribution to Φ in Eq. (1). The second term $G(x,y)$ gives the singular contribution. For $4 < \gamma \leq 5$ the function $G(x,y)$ has the following properties: $G(0,y) \propto y^4$ at small y, $d^n G(0,y)/dy^n \neq 0$ for $n \geq 4$ at $y=0$. Owing to the condition (3), the function $G(x,y)$ increases slower than y^3 at $y \gg 1$.

It is convenient to use a continuous approximation for a degree distribution $P(k) = Ak^{-\gamma}$, where A is a normalization factor. Then Eq. (1) may be rewritten in the following form:

$$\Phi(x,0) = f_2 x^2 + \varphi(x)x^4 + A\int_m^\infty \frac{dk}{k^\gamma} G(x,xk), \qquad (11)$$

where m is the smallest degree in $P(k)$, $\varphi(x)$ is a smooth function which is determined by the convergent moments $\langle k^l \rangle$ with $l \leq 3$, $\varphi(0) \neq 0$. In order to find Φ at small x, one can put $G(x,y) \approx G(0,y)$ into Eq. (11). Considering the integral over a variable $y=xk$, one can show that the region $mx \leq y \leq b$ gives a leading contribution. b is a model parameter. We obtain

$$\Phi(x,H) = -Hx + a(T-T_c)x^2 + Ax^4 \ln(b/x), \qquad (12)$$

$$\Phi(x,H) = -Hx + a(T-T_c)x^2 + Bx^{\gamma-1}, \qquad (13)$$

for $\gamma = 5$ and $3 < \gamma < 5$, respectively. A and B are constants at $T = T_c$. The critical behavior for the case $f_3 = 0$, $f_4 > 0$ at

TABLE I. Critical behavior of the order parameter x, the specific heat δC, and the response function χ_x of a model on networks with a degree distribution $P(k) \sim k^{-\gamma}$ for various values of the coefficients f_3 and f_4, and exponent γ. $\tau \equiv 1 - T/T_c$, c is a constant which is determined by the complete form of $P(k)$. In the case $f_3 < 0$, or $f_4 < 0$ if $f_3 = 0$, at $\gamma > 3$, the system undergoes a first-order phase transition.

		x	$\delta C(T < T_c)$	χ_x
$f_3 = 0,$ $f_4 > 0$	$\gamma > 5$	$\tau^{1/2}$	jump at T_c	τ^{-1}
	$\gamma = 5$	$\tau^{1/2}/(\ln \tau^{-1})^{1/2}$	$1/\ln \tau^{-1}$	
	$3 < \gamma < 5$	$\tau^{1/(\gamma-3)}$	$\tau^{(5-\gamma)/(\gamma-3)}$	
$f_3 > 0$	$\gamma > 4$	τ	τ	
	$\gamma = 4$	$\tau/(\ln \tau^{-1})$	$\tau/(\ln^2 \tau^{-1})$	
	$3 < \gamma < 4$	$\tau^{1/(\gamma-3)}$	$\tau^{(5-\gamma)/(\gamma-3)}$	
arbitrary	$\gamma = 3$	e^{-cT}	$T^2 e^{-2cT}$	T^{-1}
f_3 and f_4	$2 < \gamma < 3$	$T^{-1/(3-\gamma)}$	$T^{-(\gamma-1)/(3-\gamma)}$	T^{-2}

$\gamma > 3$ is summarized in Table I. It was observed in the Ising model on "equilibrium" uncorrelated random networks [8,9].

Note that the divergence of $\langle k^4 \rangle$ does not change the critical behavior of χ_x.

(c) $2 < \gamma \leq 3$. Now $\langle k^2 \rangle$ diverges. Equation (5) shows that in this situation there is no phase transition at any finite temperature. According to Eq. (8), a singular term $x^2 s(x)$ is expected to appear in Φ. Calculations of Φ may be carried out in a similar way as above, using the function $G(x,y) = \phi(x,y) - \Sigma_{m=1} \phi_{m1} x^m y$, Eq. (10). We find

$$\Phi(x,H) = -Hx + Cx^2 - Dx^2 \ln(b/x), \quad (14)$$

$$\Phi(x,H) = -Hx + C'x^2 - D'x^{\gamma-1} \quad (15)$$

for $\gamma = 3$ and $2 < \gamma < 3$, respectively. The coefficients C, D, C', and D' are functions of T and H. If C, D, C', and D' are positive, then at arbitrary finite temperature $T \gg 1$ in zero field $H = 0$, Eq. (6) has stable nontrivial solutions

$$x = b \exp[-(2C+D)/(2D)], \quad \chi_x = 1/(2D), \quad (16)$$

$$x = \left(\frac{2C'}{(\gamma-1)D'} \right)^{-1/(3-\gamma)}, \quad \chi_x = \frac{1}{2(3-\gamma)C'} \quad (17)$$

for $\gamma = 3$ and $2 < \gamma < 3$, respectively. As in the Landau theory, the results (16) and (17) are obtained in terms of the coefficients of the thermodynamic potential. Note that in the situation where a phase transition is absent at any finite temperature, the phenomenological theory can not determine the temperature behavior of the coefficients C, D, C', and D'.

In this situation, the temperature dependences of the coefficients can be found only by a microscopic theory (see below).

In the general case the symmetry of the model admits nonzero f_3. In the case $f_3 > 0$ the analysis of the analytical properties of Φ can be performed as above. For $\gamma > 4$, Φ is a smooth function of x and leads to the standard critical behavior. At $\gamma \leq 4$, Φ contains a singular term: (a) $x^3 \ln x$ at $\gamma = 4$, (b) $x^{\gamma-1}$ in the ranges $4 > \gamma > 3$ and $3 > \gamma > 2$, (c) $x^2 \ln x$ at $\gamma = 3$. The corresponding critical behavior for $\gamma > 3$ is represented in Table I. This behavior was observed for percolation on uncorrelated random networks [17]. In the range $2 < \gamma \leq 3$ when $\langle k^2 \rangle$ diverges, Φ has the universal form, Eqs. (14) and (15).

In the case $f_3 < 0$, or $f_4 < 0$ if $f_3 = 0$, when $\langle k^2 \rangle$ converges, a first-order phase transition occurs. In agreement with this prediction we found such a transition in the q-state Potts model with $q \geq 3$ on "equilibrium" uncorrelated random networks by use of the approach of Ref. [8]. In the limit $\gamma \rightarrow 3$ the jump of the order parameter at the transition tends to zero, and the transition transforms into the infinite order phase transition. A detailed study of the transition will be given elsewhere.

In order to complete Table I, let us discuss the temperature behavior in the case $2 < \gamma \leq 3$ within the microscopic theory of the Ising model and percolation on "equilibrium" uncorrelated random networks [7–9,17]. For this purpose we use the more general ferromagnetic q-state Potts model which at $q = 1$ and 2 is equivalent to percolation and the Ising model, respectively (see, for example, Ref. [19]). Using the approach of Ref. [8] we obtain that in the Potts model a continuous phase transition occurs at the exact critical temperature

$$T_c = 2/\ln \frac{\langle k^2 \rangle + (q-2)\langle k \rangle}{\langle k^2 \rangle - 2\langle k \rangle}. \quad (18)$$

Hereafter, we set the energy of ferromagnetic interaction between nearest neighbors $J = 1$. The parameter $p_c = 1 - \exp(-2/T_c)$ determines the percolation threshold [16,17] and establishes the relation between the Potts model and percolation.

Let us consider the case $2 < \gamma \leq 3$. Here, $\langle k^2 \rangle$ diverges. T_c is infinite for the infinite networks. In any finite network, $\langle k^2 \rangle < \infty$, and T_c is finite, although it may be very high, $T_c \cong 2\langle k^2 \rangle/(\langle k \rangle q)$. In the temperature region $T \gg 1$, where $x \ll 1$, the Potts model demonstrates universal behavior at all $q \geq 1$:

$$x \approx (q/\langle k \rangle)e^{-qT/\langle k \rangle}, \quad \chi_x \propto T^{-1}, \quad (19)$$

$$x \sim T^{-1/(3-\gamma)}, \quad \chi_x \propto T^{-2} \quad (20)$$

for $\gamma = 3$ and $2 < \gamma < 3$, respectively. Without the continuum approximation, we have instead of $\langle k \rangle$ in the exponential, a constant which is determined by the complete form of $P(k)$. In Ehrenfest's classification, this transition is of infinite order, as all temperature derivatives of Φ are finite. The results (16) and (17) agree with Eqs. (19) and (20) if we put C, C'

$\propto T^2$, $D, D' \propto T$. The exponential behavior (19) has been revealed in epidemic spreading within scale-free networks with $\gamma = 3$ [18] and in percolation on these networks [17].

At small x there is the following relationship between the response function χ_x and the susceptibility $\chi = dM/dH$: $\chi \approx 2/qT + \langle k \rangle \chi_x / q$. At $\gamma > 3$ the critical behavior of χ is determined by χ_x: $\chi \approx \langle k \rangle \chi_x / q$. At $2 < \gamma \leq 3$ we have $\chi \propto 1/qT$, as the paramagnetic contribution $2/qT$ is of the order of χ_x or larger. Let us discuss the results of theoretical studies and simulations of critical phenomena in different networks on the basis of the phenomenological theory.

The phenomenological theory as well as the Landau theory assumes that the contribution of fluctuations to the thermodynamic potential is small. Above we have shown that the theory gives the exact critical behavior of the Ising model, percolation and Potts model on "equilibrium" uncorrelated random networks [8,9,17]. The reason for this is that these networks have a local treelike structure and vertices are statistically equivalent [20–22]. Due to these properties, vertices in the networks can be regarded as forming a Bethe-lattice structure for which a mean-field approach is exact [23]. It means that the fluctuation contribution is negligibly small. Note that in a graph with a Cayley treelike structure, vertices are statistically inequivalent, and a mean-field approach is valid only for vertices deep within such a graph [23].

The origin of a deviation from the standard mean-field critical behavior is different for regular lattices and networks. In a regular lattice the deviation is caused by strong critical fluctuations which depend crucially on the space dimension. In networks the deviation is brought about by the most connected vertices which induce strong correlations in their close neighborhood. With decreasing exponent γ the relative number of highly connected vertices increases and their role turns out to be more important.

Recent simulations [7] of the Ising model on the Barabási-Albert scale-free network with the degree distribution $P(k) = Ak^{-3}$ revealed a temperature behavior described by Eq. (19) as well as in the Potts model on "equilibrium" uncorrelated random networks. Unlike the latter networks, the Barabási-Albert net is correlated. Nevertheless, results of the simulations agree with the universal temperature behavior predicted by the phenomenological theory for networks with the divergent moment $\langle k^2 \rangle$.

The exact results for percolation on small-world networks were obtained in Ref. [6]. The Ising and XY models on small-world networks were studied analytically and by use of Monte Carlo simulations [10–13]. These networks, introduced by Watts and Strogatz [24], have a Poisson-like degree distribution and large clustering coefficients. It was found that the critical behavior is characterized by the standard mean-field critical exponents. This result is in agreement with the prediction of the phenomenological theory that the standard mean-field critical behavior should occur in networks described by a degree distribution with convergent moments.

Thus, the analysis of critical behavior of the Ising, Potts, and XY models, percolation and epidemic spreading on uncorrelated random, scale-free and small-word networks shows that the critical behavior in networks depends crucially on the form of the degree distribution and the symmetry underlying a model in a complete agreement with the phenomenological theory. The studied networks differ in clustering coefficients, degree correlations, etc. However, it looks as though these characteristics are not relevant for critical behavior. Their role in the formation of the critical exponents is still unclear. Further investigations in this topic are necessary. It would be interesting to find a network where critical behavior differs from a mean-field one due to strong fluctuations. However, even in this case it is expected that when T tends to T_c, a temperature region of the mean-field critical behavior precedes a region of strong fluctuations.

Real systems such as the Internet, WWW, or biological networks have a network structure with a power-law degree distribution with exponent γ below 3, see Refs. [3,4]. The phenomenological theory predicts for this case the power-law critical behavior (20), see also the last line in Table I. This behavior agrees with an empirical observation [5] for the nd.edu domain of the WWW, where the variations of the size of the giant component under random damage were studied. This size and the number of damaged vertices play a role of the order parameter x and the control parameter, respectively.

In conclusion, the phenomenological theory shows that the deviation of the critical behavior of interacting systems with a network structure from the standard mean-field behavior emerges when a degree distribution has divergent moments. The theory predicts different classes of critical behavior in agreement with microscopic studies of the Ising and Potts models, and percolation on "equilibrium" uncorrelated random networks. It also agrees with results previously obtained for various models on small-world and evolving networks. The theory can easily be generalized for models with a multicomponent order parameter and can give useful insight into collective effects in different networks discussed in connection with the Internet, biological networks, etc. Using this approach, one can also study the critical relaxation in models on networks.

---

[1] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).

[2] S. H. Strogatz, Nature (London) **401**, 268 (2001).

[3] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).

[4] S. N. Dorogovtsev and J. F. F. Mendes, Adv. Phys. **51**, 1079 (2002).

[5] R. Albert, H. Jeong, and A.-L. Barabási, Nature (London) **406**, 378 (2000).

[6] C. Moore and M. E. J. Newman, Phys. Rev. E **61**, 5678 (2000);

62, 7059 (2000); M. E. J. Newman, I. Jensen, and R. M. Ziff, *ibid.* **65**, 021904 (2002).

[7] A. Aleksiejuk, J. A. Holyst, and D. Stauffer, Physica A **310**, 260 (2002).

[8] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Phys. Rev. E **66**, 016104 (2002).

[9] M. Leone, A. Vázquez, A. Vespignani, and R. Zecchina, Eur. Phys. J. B **28**, 191 (2002).

[10] M. Gitterman, J. Phys. A **33**, 8373 (2000).

[11] A. Barrat and M. Weigt, Eur. Phys. J. B **13**, 547 (2000).

[12] B. J. Kim, H. Hong, P. Holme, G. S. Jeon, P. Minnhagen, and M. Y. Choi, Phys. Rev. E **64**, 056135 (2001).

[13] H. Hong, B. J. Kim, and M. Y. Choi, Phys. Rev. E **66**, 018101 (2002).

[14] G. Bianconi, Phys. Lett. A **303**, 166 (2002).

[15] R. Cohen, K. Erez, D. ben-Avraham, and S. Havlin, Phys. Rev. Lett. **85**, 4626 (2000).

[16] D. S. Callaway, M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. Lett. **85**, 5468 (2000).

[17] R. Cohen, D. ben-Avraham, and S. Havlin, Phys. Rev. E **66**, 036113 (2002).

[18] R. Pastor-Satorras and A. Vespignani, Phys. Rev. Lett. **86**, 3200 (2001); Phys. Rev. E **63**, 066117 (2001).

[19] F. Y. Wu, Rev. Mod. Phys. **54**, 235 (1982).

[20] A. Bekessy, P. Bekessy, and J. Komlos, Stud. Sci. Math. Hungary **7**, 343 (1972); E. A. Bender and E. R. Canfield, J. Comb. Theory, Ser. A **24**, 296 (1978); B. Bollobás, Eur. J. Comb. **1**, 311 (1980); N. C. Wormald, J. Comb. Theory, Ser. B **31**, 156, 168 (1981).

[21] M. Molloy and B. Reed, Random Struct. Algorithms **6**, 161 (1995).

[22] M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. E **64**, 026118 (2001).

[23] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982).

[24] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).